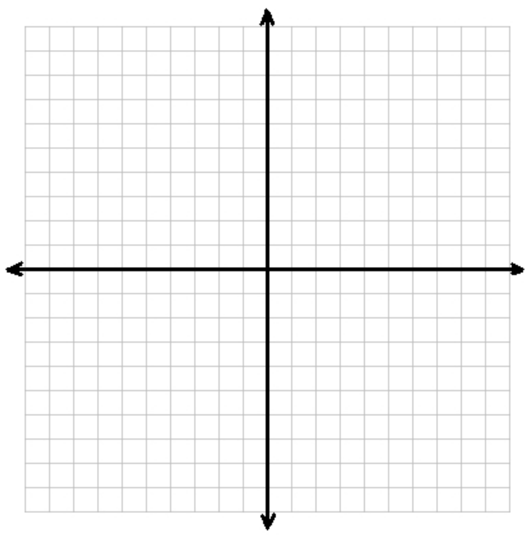
MATH 1650: SECTION 1.2: CONSTANT AND LINEAR FUNCTIONS

DEFINITION: A **constant function** is a function of the form: $f(x) = b$ with domain $(-\infty, \infty)$.

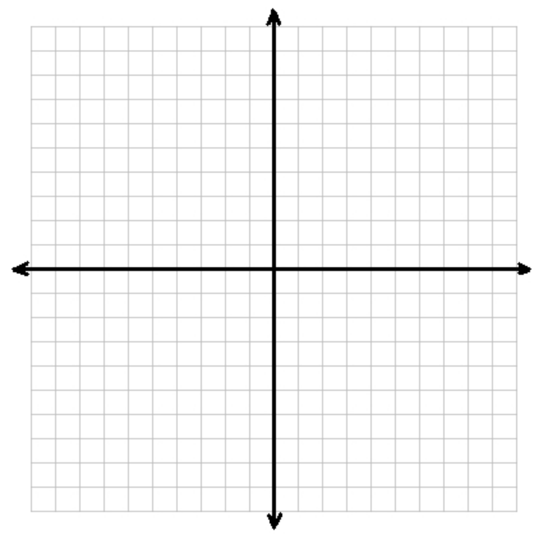
NOTE: Constant functions are called 'constant' since the output never changes.

EXAMPLE: Graph the following functions.

- $f(x) = 6$



- $g(x) = 0$

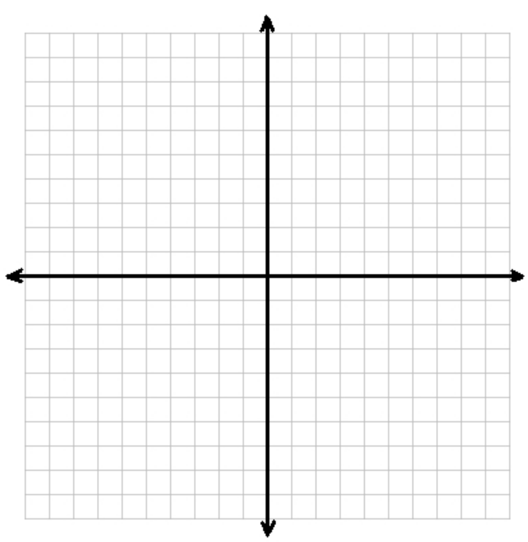


DEFINITION: A **linear function** is a function of the form: $f(x) = mx + b$, $m \neq 0$ with domain $(-\infty, \infty)$.

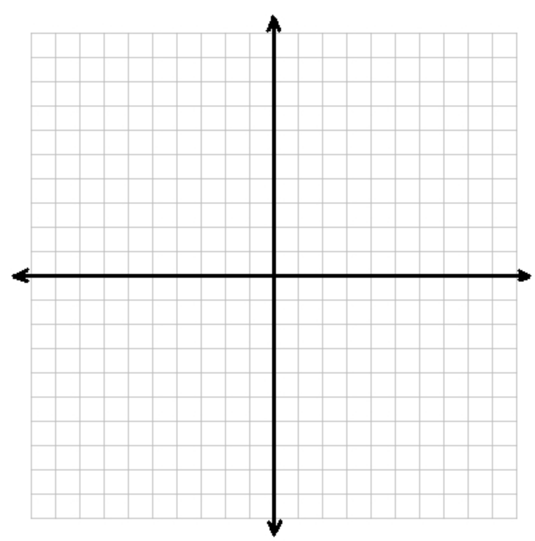
NOTE: $f(x) = mx + b$ is the **slope-intercept** form of a linear function which is useful in graphing linear functions.

EXAMPLE: Graph the following functions.

- $F(x) = 3x - 2$



- $G(x) = \dfrac{3 - x}{2}$



INTERCEPTS: Consider the graph of a function where $y = f(x)$.

- The point where the graph intercepts the y-axis is called the 'y-intercept' of the graph.

  NOTE: To find the y-intercept, plug in $x = 0$. Since $y = f(x)$ when $x = 0$, $y = f(0)$, so the y-intercept is $(0, f(0))$.

- The points where the graph intercepts the x-axis are called the 'x-intercepts' of the graph.

  NOTE: To find the x-intercepts, solve $y = f(x) = 0$. The x-intercepts then have the form $(x, 0)$.

SLOPE AS A RATE OF CHANGE:

We know that the slope of the line containing two points (x_0, y_0) and (x_1, y_1) is given by:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0},$$

provided $x_0 \neq x_1$. (If $x_0 = x_1$, we'd be trying to divide by 0 . . .)

On the graph of a function f, $y = f(x)$, so we can rewrite the slope formula using function notation as follows. Given two different inputs x_0 and x_1, we call $y_0 = f(x_0)$ and $y_1 = f(x_1)$. The slope equation becomes:

$$m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta[f(x)]}{\Delta x} = \frac{\text{change in outputs}}{\text{change in inputs}}$$

EXAMPLE: The function $T(t) = -2t + 10$ gives the temperature (in °F) on a particular day t hours after 4 PM.

- Find $T(0)$ and interpret what this means in terms of time and temperature.

- Find the slope of $T(t)$ and interpret what this means in terms of time and temperature.

- When will the temperature reach 0°F?

Recall the **point-slope** form of a line: $y = m(x - x_0) + y_0$, $y = f(x)$ gives the **point-slope** form of a linear function:

$$f(x) = m(x - x_0) + f(x_0)$$

The point-slope form of a linear function is very helpful when building a linear function.

EXAMPLE: Suppose a linear function f satisfies $f(-1) = 2$ and $f(3) = 0$.

- Find the slope of the linear function: $m = \dfrac{\Delta[f(x)]}{\Delta x} = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} = \dfrac{f(3) - f(-1)}{3 - (-1)} =$

- Use the point-slope formula of a linear function to find a formula for $f(x)$:
  - Using $x_0 = -1$ and $f(x_0) = f(-1) = 2$:

    $f(x) = m(x - x_0) + f(x_0) =$

  - Using $x_0 = 3$ and $f(x_0) = f(3) = 0$:

    $f(x) = m(x - x_0) + f(x_0) =$

  - You should have gotten the same answer (once simplified!) regardless of which x value you chose to be x_0. Does this surprise you?

PIECEWISE-DEFINED FUNCTIONS: Consider the function: $f(x) = \begin{cases} 6 - x & \text{if} \quad x \leq 2 \\ 2x - 1 & \text{if} \quad x > 2 \end{cases}$

Notice we have **two** formulas for $f(x)$ depending on what the input is:

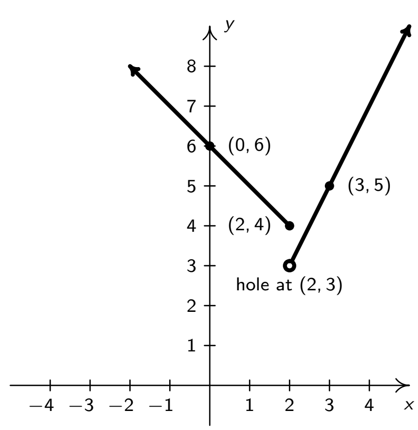
- if $x \leq 2$, then we use the formula $f(x) = 6 - x$.
- if $x > 2$, then we use the formula $f(x) = 2x - 1$.

Find the following function values:

- $f(0)$
- $f(1)$
- $f(2)$
- $f(2.1)$
- $f(3)$
- $f(4)$

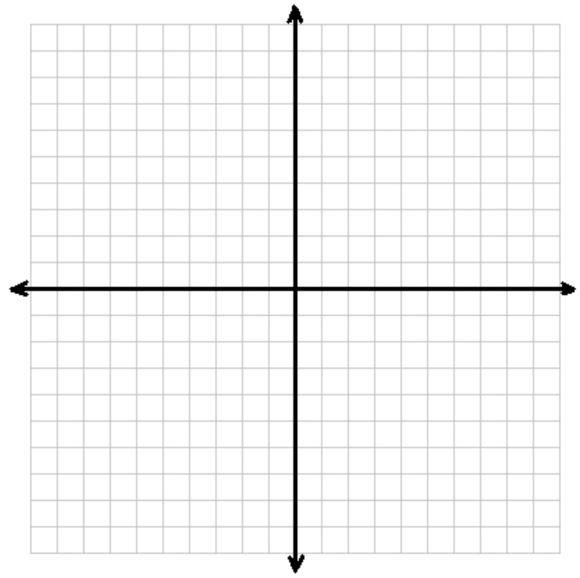
Let's graph $y = f(x)$. Just like the formula for $f(x)$ has two parts, the graph will have two parts:

- First, we need to graph the **part** of the line $y = 6 - x$ which corresponds to $x \leq 2$.
  Remember, we need only **two** points to graph a line, so we need to choose two x-values where $x \leq 2$.
  We choose $x = 0$ and find $y = 6 - 0 = 6$ which corresponds to the point $(0, 6)$. (The y-intercept!)
  Next we choose $x = 2$ because this is endpoint of the interval $x \leq 2$.
  We get $y = 6 - 2 = 4$ which corresponds to the point $(2, 4)$.
  So our first part of the graph is a line containing $(0, 6)$ and extending to and stopping at $(2, 4)$, including $(2, 4)$.
- Next we need to graph the **part** of the line $y = 2x - 1$ for $x > 2$.
  Once again we need only plot two points to graph this line.
  If we choose the endpoint $x = 2$, we would get $y = 2(2) - 1 = 3$ corresponding to the point $(2, 3)$.
  However since the formula $y = 2x - 1$ is valid for $x < 2$ but not when $x = 2$, we do not plot a **point** at $(2, 3)$.
  Instead, we draw an 'open circle,' '$\circ$' or 'hole' in the graph at $(2, 3)$.
  Next, we pick a value of $x > 2$, say $x = 3$ and find $y = 2(3) - 1 = 5$ and plot the point $(3, 5)$.
  Hence, the second part of the graph is a line starting at the hole at $(2, 3)$ and extending to the **right** containing $(3, 5)$.

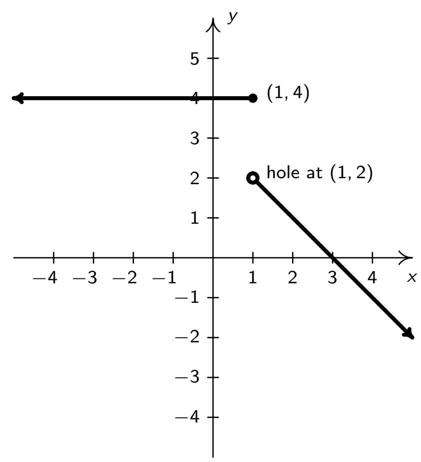


The graph of $f(x) = \begin{cases} 6 - x & \text{if} \quad x \leq 2 \\ 2x - 1 & \text{if} \quad x > 2 \end{cases}$

EXAMPLE: Graph $f(x) = \begin{cases} x+3 & \text{if} \quad x < -1 \\ 2-x & \text{if} \quad x \geq -1 \end{cases}$



EXAMPLE: Find a formula for the function C graphed below:



$y = C(x)$.

GENERAL FUNCTION BEHAVIOR: We know from our experience with lines that:

- The y-values on $y = b$ are constant on $(-\infty, \infty)$. That is, the y-values do not change.
- The y-values on $y = mx + b$ are:
  - increasing on $(-\infty, \infty)$ if $m > 0$. That is, the y-values rise as we move from right to left if the slope is positive.
  - decreasing on $(-\infty, \infty)$ if $m < 0$. That is, the y-values fall as we move from right to left if the slope is negative.

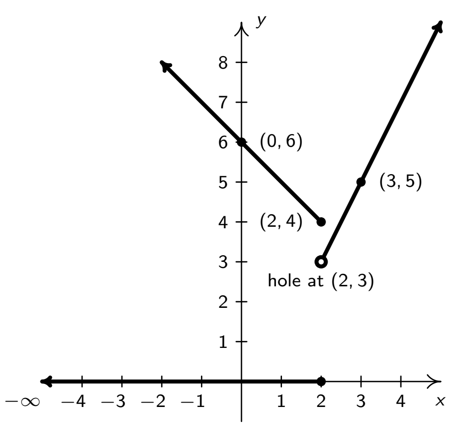
We generalize these ideas to more general functions with the following definitions.

DEFINITIONS: Let f be a function defined on an interval I. Then f is said to be:

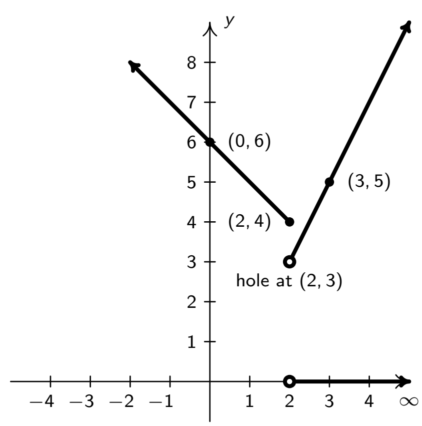
- **constant** on I if $f(a) = f(b)$ for all a, b in I. (i.e., outputs don't change with inputs; outputs are **constant**.)
  NOTE: The graph of a function that is constant over an interval is a horizontal line.
- **increasing** on I if whenever $a < b$, then $f(a) < f(b)$. (i.e., as inputs increase, outputs **increase**.)
  NOTE: The graph of a function that is increasing over an interval **rises** as one moves from left to right.
- **decreasing** on I if whenever $a < b$, then $f(a) > f(b)$. (i.e., as inputs increase, outputs **decrease**.)
  NOTE: The graph of a function that is decreasing over an interval **falls** as one moves from left to right.

NOTE: When we ask 'where' a function is increasing (or decreasing or constant), we are looking for a **intervals of inputs**.

EXAMPLE: $f(x) = \begin{cases} 6 - x & \text{if} \quad x \leq 2 \\ 2x - 1 & \text{if} \quad x > 2 \end{cases}$:

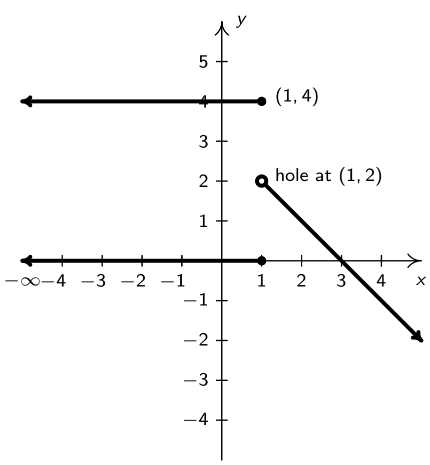


f is decreasing on $(-\infty, 2]$

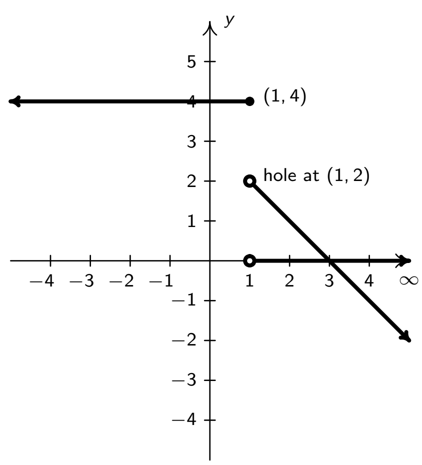


f is increasing on $(2, \infty)$

EXAMPLE: $y = C(x)$:



C is constant on $(-\infty, 1]$



C is decreasing on $(1, \infty)$.

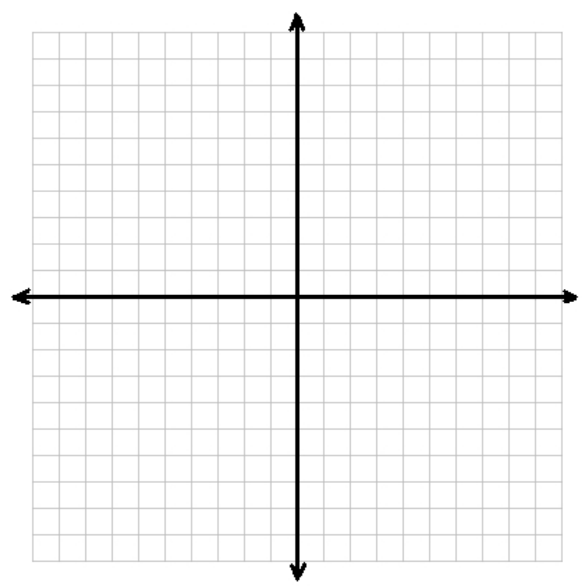
MATH 1650 APPLICATIONS OF LINEAR FUNCTIONS

1. The cost C (in dollars) to produce x "I'd rather be a Sasquatch" T-Shirts is $C(x) = 2x + 26$, $x \geq 0$.

   (a) Find and interpret $C(0)$.

   (b) How many shirts can be produced on a budget of \$117?

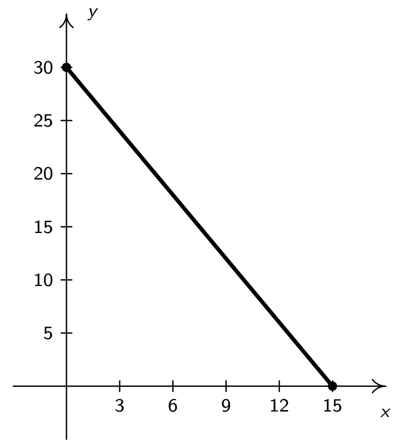
   (c) Graph $y = C(x)$.



   (d) Find and interpret the slope of this graph in terms of the cost of shirts and the number of shirts made.

   (e) Is C increasing, decreasing, or constant? What does this say about the cost of shirts?

2. The price p (in dollars per shirt) that is charged for each "I'd rather be a Sasquatch" T-shirt is a function of how many shirts are sold, x. The graph of p is below.



$$y = p(x).$$

(a) Find the domain and range of p.

domain:

range:

(b) Is p increasing, decreasing or constant? What does this say about the price of shirts?

(c) Find and interpret the slope of this graph in terms of the price per shirt and the number of shirts sold.

(d) Find an expression for $p(x)$.

(e) Solve and interpret $p(x) = 12$.

3. At 8 AM, it is 45° F. At noon, it is 53°F.

   (a) Find a linear function which models the temperature, $T(t)$ as a function of the number of hours after 6 AM, t.

   HINT: If t is the number of hours **after** 6 AM, 8 AM corresponds to $t = 2$ and noon corresponds to $t = 6$ ...

   (b) Interpret the slope of your function in terms of time and temperature.

   (c) Use this model to predict the temperature at 1 PM.

4. A popular iPhone carrier offers the following data plan: use any amount of data up to and including 2 Gigabytes for \$10 per month with an 'overage' charge of \$15 per Gigabyte. Write a piecewise function which computes the cost, $C(G)$, in dollars, of using G gigabytes of data.

USING SLOPE TO ANALYZE NON-LINEAR FUNCTIONS: AVERAGE RATE OF CHANGE

DEFINITION: If f is a function defined on an interval $[x_0, x_1]$, the **average rate of change of** f is given by:

$$ARC_{[x_0,x_1]} = \frac{\Delta[f(x)]}{\Delta x} = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\text{change in outputs}}{\text{change in inputs}}$$

NOTE: The average rate of change of f over $[x_0, x_1]$ is the slope of the line through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$. Since the average rate of change is a slope, it shouldn't be too surprising to learn that:

- The average rate of change of a **constant** function, $f(x) = b$ is always 0.
- The average rate of change of a **linear** function, $f(x) = mx + b$ is always m.

In other words, constant and linear functions are precisely those functions which have a constant rate of change.

EXAMPLE: The formula $s(t) = -5t^2 + 100t$ for $0 \leq t \leq 20$ gives the height, $s(t)$, measured in feet, of a model rocket above the Moon's surface as a function of the time after lift-off, t, in seconds.

- Find and interpret $s(0)$, and $s(10)$, and $s(20)$.

$s(0) = -5(0)^2 + 100(0) =$

$s(10) = -5(10)^2 + 100(10) =$

$s(20) = -5(20)^2 + 100(20) =$

- Find the average rate of change of s over the intervals $[0, 10]$ and $[10, 20]$.

$$ARC_{[0,10]} = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(10) - s(0)}{10 - 0} =$$

$$ARC_{[10,20]} = \frac{\Delta[s(t)]}{\Delta t} = \frac{s(20) - s(10)}{20 - 10} =$$

- Interpret each rate of change in terms of the rocket's journey.

HINT: $ARC = \dfrac{\Delta[s(t)]}{\Delta t} = \dfrac{\text{change in altitude (feet)}}{\text{change in time (seconds)}}$

$ARC_{[0,10]}$ means:

$ARC_{[10,20]}$ means:

HOMEWORK: Section 1.2: 1 - 59 odd.